1. Interest Rates

# Introduction

# Quoting Conventions

Most of us realize that when we take out a mortgage in order to purchase a house the interest rate quoted is for one year. If the life of the mortgage is 25 years the actual amount repaid will be many multiples of the initial amount borrowed. We say rates are quoted on an annualized basis. The annualized rate needs to be scaled by time in order to obtain the actual amount paid or earned over the life of any given financial product.

The way in which an annualized rate quote is scaled with time depends on the compounding frequency. Compounding means earning interest on interest. In the simplest scenario there is no compounding and the rate paid or earned is simply the linear product of the annualized rate and the length of time. We call such a rate a simple rate. If we let



be the simple rate,



be the principle amount borrowed/lent and

,

be the length of time in years over which the principle is lent or borrowed, then the following expression gives the total amount repaid

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If we borrowed $100 for two years and the simple interest rate was 5% per annum, then at the end of the two years we would have paid back

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Most rates, however, are not quoted using simple interest. Instead the interest compounds and the borrower pays interest on interest. Consider the situation where a rate quote with annual compounding is specified as 5%. At the end of one year the borrower owes **** For the second year interest is calculated against this new notional of 105 so that at the end of the second year the total repaid is **** We can rewrite the expression for the total amount repaid over the two years as

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Compounding frequencies of semi-annual or quarterly are common in finance and in full generality the rate expression for a rate quoted with compounding m times per annum is given as

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If we increase the compounding frequency without bound, we get the continuously compounded rate.

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The continuously compounded rate e is the most important format for quantitative finance as it simplifies many calculations.

**ToDo:** Add an appendix showing the derivation of e.

In financial engineering we often want to be able to use a standard format that is quotation convention agnostic. The most commonly used format in quantitative finance is the discount factor. It gives us an exact multiplicative factor to be used to discount cash flows. To convert from a rate expression such as the ones given above we use

So to convert from a given discount factor to these rates we solve algebraically from the expression

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## Converting between compounding frequencies

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. They are two different units of measurement. We can convert between rates with different compounding frequencies, say m and by solving the equation.



### Derivation

 Taking natural logarithms of each side

 Note that 

 Divide each side by mt

 Note that 

 Taking exponents

 Subtract one from each side

 Multiply both sides by m



## Effective rates

Often rates with different compounding frequencies are converted to equivalent annualized rates for comparison (Note here the annualized refers to the compounding frequency. All rates are quoted on an annualized basis). This equivalent annualized rate is known as the effective rate. The effective rate is also sometimes known as the annualized percentage rate.

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## Zero Rates

The zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for n years. All interest and principal is earned at the end of n years. If a five year zero rate with continuous compounding is quoted as 5% per annum. This means that $100, if invested for 5 years grows to



Zero treasury rates can be calculated in two ways

* Observe the yield on strips
* Bootstrap from treasury bills and bonds

# Day Count Conventions

The day count defines the way interest accrues over time. More specifically, given an annualized rate and two dates the day count defines the year fraction that will be applied to the rate expression to obtain a growth/discount factor.

# Forward Interest Rates

A forward interest rate is the rate agreed now that will apply to borrowing/lending over some period in the future. So instead of borrowing for one year now, we might enter into a forward loan to borrow cash for one year starting in one year’s time. A forward loan will then create cash flows as shown.

Pricing forward interest rates

X

X

T= 1 year

I

T= 1 year

The interesting thing about forward rates is that we can calculate them from existing spot rates. Why should this be? Using arbitrage arguments we can see that borrowing money for two years spot is the same as borrowing money for one year spot and entering into a one year forward loan. Put another way, borrowing money for one year forward is the same as borrowing money for two years and investing it for one year. The forward rate from to must be set such that no-one can make money by buying/selling interest rate instruments in the cash market and taking offsetting positions in interest rate forwards. The key to ensuring this is to make sure that



This relationship only holds where the rates are quoted with continuously compounding



From the laws of powers



Taking the Napier log of both sides



# The Yield Curve

In finance, it is common for instruments of different maturities to provide different rates. In order to compare the returns for different maturities they are gathered into a collection knows as a yield curve. If we plot the yield against maturity a simple yield curve might look like the following.

## Properties of Yield Curves

### Slope of Curve

* Long dated rates, in some respects, reflect expectations for future short rates. Imagine I want to invest for ten years. I could invest for ten years or invest for five years and then another five years. If current five year rate is 5% and the ten year rate is 7.5% that that implies that the expectation is the five year rate will rise to 10% in five years time
* In general, a yield curve should slope upwards with longer dated maturities having higher yields than shorter dated maturities.
* If short dated maturities have higher rates than long dated maturities the curve is said to be inverted
* An inverted yield curve means the market expects short-term rates to decrease. Investors are hence willing to accept a lower rate for long-term investments.

### How the BoE Controls Base Rates

* The BoE ensures the rates MPC decides are the rates banks pay each other
* 59 banks and financial institutions have a reserve account with BoE
* The reserve account is used to handle transactions between the member banks
* At the start of the month each member makes an estimate of the amount of money it will hold in the account
* This is the biggest difference the bank will see on any given day between the amount it receives and the amount it pays out ( aggregate of millions of customer transactions)
* The penalties for unauthorised overdraft are sever so banks typically borrow from other banks late in the day and pay interest at the overnight rate
* If that doesn’t work the bank can borrow from the emergency fund which charges one percent over the base rate
* If the overnight rates are above the base rate banks will chose to lend in those markets rather than holding funds in the reserve account flooding the market with liquidity
* If the overnight rates are under the base rate banks will hold excess fund in the reserve account until rates increase

Term

Annualized Continuously Compounded zero rates



In the same way as u

# Yield Curves Interpolation

## Straight Line

### General Equation



The gradient, slope or rate of change

The y-intercept

### Equation of the Straight Line through two points

From the general form of the equation we can derive an equation of the straight line through two given points by noting that the gradient m is also the derivative of rate of change.



Second note that the offset is the difference between the y value of one of the points and the product of the derivative and the corresponding x-value



Substituting these into the original we get









The gradient, slope or rate of change

The difference between the product of x co-ordinate and the derivative and the actual y co-ordinate gives the y-offset

We can also re-write this equation as



And re-arrange to get



And once more



And again



Linear Interpolation

Given two bracketing pointsand we can linearly interpolate between then as follows



## Constant Forward Yield Interpolation

Given a set of continuous rates we can compute consistent forward rates such that arbitrage is not possible as follows.

Term

Annualized Continuously Compounded zero rates



First we note that for continuously compounded rates the following must hold. The forward rate from to must be set such that no-one can make money by buying/selling interest rate instruments in the cash market and taking offsetting positions in interest rate forwards. The key to ensuring this is to make sure that



The reason is that with continuously compounded rates



From the laws of powers



Taking the Napier log of both sides



So how do we use this relation to interpolate of a continuously compounded yield curve? First we calculate the forward rate between the two nearest pillars as mentioned previously



### 2. Use the forward rate to interpolate the yield

Now that we know the forward rate we calculate the interpolated rate by adding the rate to a proportion of the forward from to . We now get



Finally we multiply this rate by 1/t to get back to an annualized rate



# Calculations

## Calculating returns from asset levels

If an asset price grows from  to some new amount we can calculate the annualized, continuously compounded return as



## Calculating the year fraction from a df and a zero rate

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We now take the Napier logarithm of both sides to get

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From the properties of logarithms we know that ****

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# Questions

1. Convert a discount factor df to a **continuous** rate?

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1. Convert a discount factor df to a semi-annual rate?

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1. Convert a discount factor df to a simple rate?

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1. Convert discount factor to a rate that compounds m times per annum?

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1. Convert a continuous rate to a discount factor?

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1. Convert a simple rate to a discount factor?

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1. Convert a rate compounding m times per annum to a discount factor?

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1. Given a sensitivity of a product to a change in continuous rates derive an expression for the sensitivity of the product to a change in annualized rates?

Let

* P = price of the product
* C = the continuous rate
* A = the annualized rate
* 

Now by the chain rule we know

Now we know that so we get therefore





1. Obtain a simple forward rate from a continuously compounded yield curve?

Term

Annualized Continuously Compounded zero rates



There are multiple ways of deriving this.

### 1. Calculate discount factors for 18months and 24months





We note that to prevent arbitrage that

 and since then





Now convert the discount factor to a simple rate by noting that





### 2. Use the forward relation